Definite Programs
Herbrand-Interpretations

The true atoms of the Herbrand base correlate with the corresponding interpretation.
Proposition:
Let $P$ be a definite program and $\{M_i\}_{i \in I}$ a non-empty set of Herbrand models for $P$. Then $\bigcap_{i \in I} M_i$ is an Herbrand model for $P$. If $\{M_i\}_{i \in I}$ contains all Herbrand models for $p$, then $M_P := \bigcap_{i \in I} M_i$ is the least Herbrand model for $P$. 
Idea

- \((2^{BP}, \subseteq)\) is a lattice of all Herbrand interpretations of \(P\) with the bottom element \(\emptyset\) and the top element \(BP\).

- The least upper bound (lub) of a set of interpretations is the union, the greatest lower bound is the intersection.
Proposition:
Let $P$ be a definite program and $\{M_i\}_{i \in I}$ a non-empty set of Herbrand models for $P$. Then $\bigcap_{i \in I} M_i$ is an Herbrand model for $P$. If $\{M_i\}_{i \in I}$ contains all Herbrand models for $p$, then $M_P := \bigcap_{i \in I} M_i$ is the least Herbrand model for $P$.

Proof:
$\bigcap_{i \in I} M_i$ is an Herbrand interpretation. Show, that it is a model. Each definite program has $B_P$ as model, hence $I$ is not empty and one can show that $M_P$ is a model.

Idea:
$M_P$ is the „most natural“ model for $P$. 
Theorem:
Let $P$ be a definite program. Then $M_P = \{A \in B_P : A \text{ is a logical consequence of } P\}$.

Proof:
$A$ is a logical consequence of $P$ iff $P \cup \{\neg A\}$ is unsatisfiable iff $P \cup \{\neg A\}$ has no Herbrand model iff $A$ is true wrt all Herbrand models of $P$ iff $A \in M_P$. 
Properties of Lattices

Definition

Let $L$ be a lattice and $T:L \rightarrow L$ a mapping. $T$ is called \textit{monotonic}, if $x \leq y$ implicates, that $T(x) \leq T(y)$. 
Properties of Lattices

Definition
Let $L$ be a lattice and $X \subseteq L$, $X$ is called directed, if each finite sub-set of $X$ has an upper bound in $X$.

Definition
Let $L$ be a lattice and $T:L \rightarrow L$ a mapping. $T$ is called continuous, if for each directed subset $X$ $T(\text{lub}(X))=\text{lub}(T(X))$. 

**Definition**

Let P be a definite program. The mapping \( T_P : 2^{BP} \rightarrow 2^{BP} \) is defined as follows: Let I be an Herbrand interpretation. Then

\[
T_P(I) = \{ A \in BP : A \leftarrow A_1 \land \ldots \land A_n \text{ is a ground instance of a clause in P and } \{A_1, \ldots, A_n\} \subseteq I \} 
\]
Practice

Let $P$ be a definite program.

\[
\begin{align*}
\text{even}(f(f(x))) & \leftarrow \text{even}(x). \\
\text{odd}(f(x)) & \leftarrow \text{even}(x). \\
\text{even}(0). 
\end{align*}
\]

Let $I_0 = \emptyset$. 

Then $I_1 = T_P(I_0) = ?$

\[
T_P(I) = \{ A \in B_P : A \leftarrow A_1 \land \ldots \land A_n \text{ is a ground instance of a clause in } P \text{ and } \{A_1, \ldots, A_n\} \subseteq I \}
\]
Example
Let P be a definite program.

```
even(f(f(x))) ← even(x).
even(0).
```

Let $I_0 = \emptyset$.

Then

- $I_1 = T_P(I_0) = \{\text{even}(0)\}$,
- $I_2 = T_P(I_1) = \{\text{even}(0), \text{even}(f(f(0)))\}$,
- $I_3 = T_P(I_2) = \{\text{even}(0), \text{even}(f(f(0))), \text{even}(f(f(f(f(0))))), ... \}$

$T_P(I) = \{A \in B_P : A ← A_1 ∧ ... ∧ A_n$

is a ground instance of a clause in P and

$\{A_1, ..., A_n\} \subseteq I}$

$T_P$ is monotonic.
Let $P$ be a definite program.

\[
\begin{align*}
\text{plus}(x, f(y), f(z)) & \leftarrow \text{plus}(x, y, z) \\
\text{plus}(f(x), y, f(z)) & \leftarrow \text{plus}(x, y, z) \\
\text{plus}(0, 0, 0) & \\
\end{align*}
\]

Let $I_0 = \emptyset$. Then $I_1 = T_P(I_0) = ?$

$T_P(I) = \{ A \in B_P : A \leftarrow A_1 \land \ldots \land A_n \}
\text{is a ground instance of a clause in } P \text{ and }
\{A_1, \ldots, A_n\} \subseteq I\}$
Practice

Let $P$ be a definite program.

\[ \begin{align*}
\text{plus}(x,f(y),f(z)) & \leftarrow \text{plus}(x,y,z) \\
\text{plus}(f(x),y,f(z)) & \leftarrow \text{plus}(x,y,z) \\
\text{plus}(0,0,0) & 
\end{align*} \]

Let $I_0 = \emptyset$.

Then

\[ \begin{align*}
I_1 &= T_P(I_0) = \{\text{plus}(0,0,0)\} \\
I_2 &= T_P(I_1) = \{\text{plus}(0,0,0), \\
& \quad \text{plus}(f(0),0,f(0)), \\
& \quad \text{plus}(0,f(0),f(0))\} \\
I_3 &= T_P(I_2) = \{\text{plus}(0,0,0), \\
& \quad \text{plus}(f(0),0,f(0)), \\
& \quad \text{plus}(0,f(0),f(0)), \\
& \quad \text{plus}(f(f(0)),0,f(0)), \\
& \quad \text{plus}(0,f(f(0)),f(0)), \\
& \quad \text{plus}(f(0),f(0),f(0)), \\
& \quad \text{plus}(f(0),f(0),f(f(0))))\} \ldots
\end{align*} \]
Fixpoint-Model

Proposition

Let $P$ be a definite program and $I$ be an Herbrand interpretation of $P$. Then $I$ is a model for $P$ iff $T_P(I) \subseteq I$.

$\implies \checkmark$

Let $I$ be an Herbrand interpretation, which is not a model of $P$ and $T_P(I) \subseteq I$. Then there exist ground instances $\{\neg A, A_1, \ldots, A_n\} \in I$ and a clause $A \leftarrow A_1 \land \ldots \land A_n$ in $P$. Then $A \in T_P(I) \subseteq I$. Refutation. $\checkmark$
Definition

let L be a complete lattice and \( T:L \to L \) be monotonic. Then we define

\[
T^\uparrow 0 = \bot \\
T^\uparrow \alpha = T(T^\uparrow (\alpha - 1)), \text{ if } \alpha \text{ is a successor ordinal} \\
T^\uparrow \alpha = \text{lub}\{T^\uparrow \beta : \beta < \alpha\} \text{ if } \alpha \text{ is a limit ordinal}
\]

\[
T^\downarrow 0 = \top \\
T^\downarrow \alpha = T(T^\downarrow (\alpha - 1)), \text{ if } \alpha \text{ is a successor ordinal} \\
T^\downarrow \alpha = \text{lub}\{T^\downarrow \beta : \beta < \alpha\} \text{ if } \alpha \text{ is a limit ordinal}
\]

Example
Theorem
Let $P$ be a definite program.
Then $M_P = \text{lfp}(T_P) = T_P^{\uparrow \omega}$, where
$T^{\uparrow \alpha} = T(T^{\uparrow ((\alpha - 1))})$, if $\alpha$ is a successor ordinal
$T^{\uparrow \alpha} = \text{lub}\{T^{\uparrow \beta} : \beta < \alpha\}$, if $\alpha$ is a limit ordinal

Proof
$M_P = \text{glb}\{I : I \text{ is an Herbrand model for } P\}$
$= \text{glb}\{I : T_P(I) \subseteq I\}$
$= \text{lfp}(T_P)$ (not shown here)
$= T_P^{\uparrow \omega}$. 
**Answer**

**Definition**

Let $P$ be a definite program and $G$ a definite goal. An *answer* for $P \cup \{G\}$ is a substitution for variables of $G$.

**Definition**

Let $P$ be a definite program, $G$ a definite goal $\leftarrow A_1 \land \ldots \land A_n$ and $\theta$ an answer for $P \cup \{G\}$. We say that $\theta$ is a *correct answer* for $P \cup \{G\}$, if $\forall((A_1 \land \ldots \land A_n)\theta)$ is a logical consequence of $P$. The answer „no“ is correct if $P \cup \{G\}$ is satisfiable.

**Example**
Theorem

Let P be a definite program and G a definite goal
← A₁ ∧ … ∧ Aₙ. Suppose θ is an answer for P ∪ {G}, such that (A₁ ∧ 
… ∧ Aₙ)θ is ground. Then the following are equivalent:

1. θ is correct
2. (A₁ ∧ … ∧ Aₙ)θ is true wrt every Herbrand model of P
3. (A₁ ∧ … ∧ Aₙ)θ is true wrt the least Herbrand model of P.
Proof
it suffices to show that 3 implies 1
(A₁ \land ... \land Aₙ)θ is true wrt the least Herbrand model of P implies
(A₁ \land ... \land Aₙ)θ is is true wrt all Herbrand models of P implies
\neg(A₁ \land ... \land Aₙ)θ is false wrt all Herbrand models of P implies
P \cup \{\neg(A₁ \land ... \land Aₙ)θ\} has no Herbrand models
implies P \cup \{\neg(A₁ \land ... \land Aₙ)θ\} has no models.