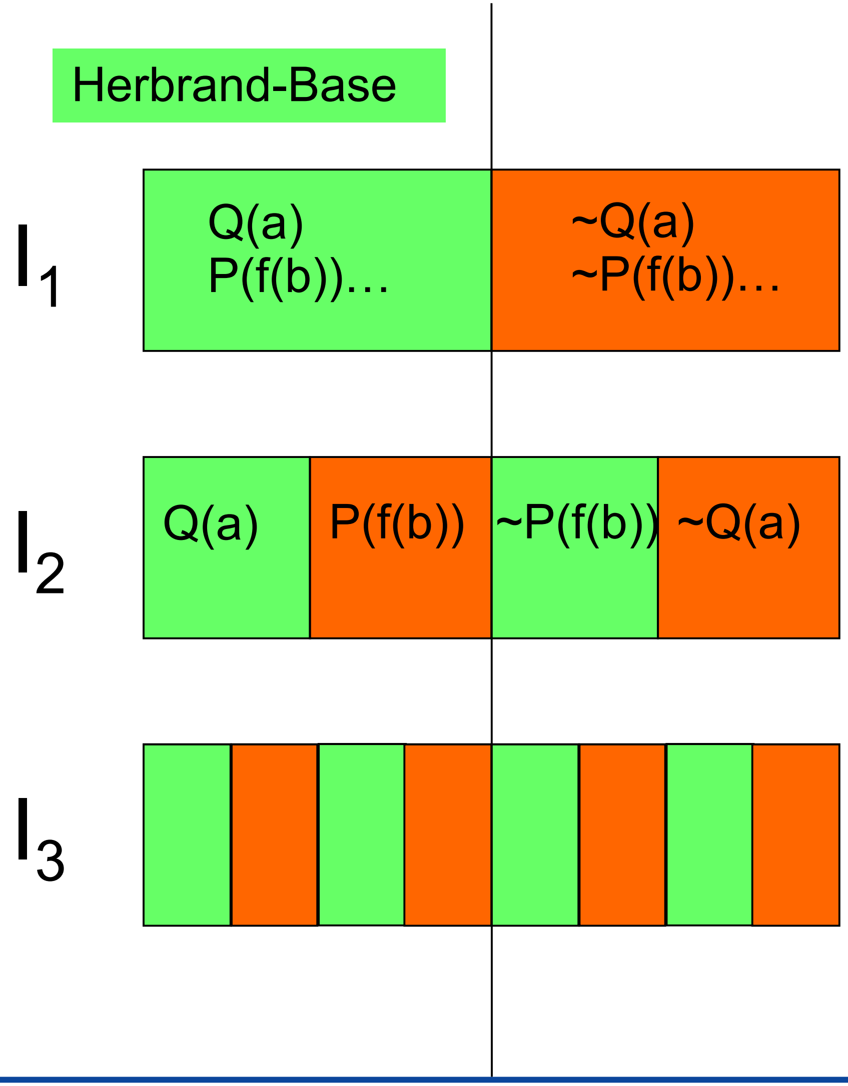
Models of Definite Programs

Herbrand-Interpretations



The true atoms of the Herbrand base correlate with the corresponding interpretation.

Model Intersection Property

Proposition:

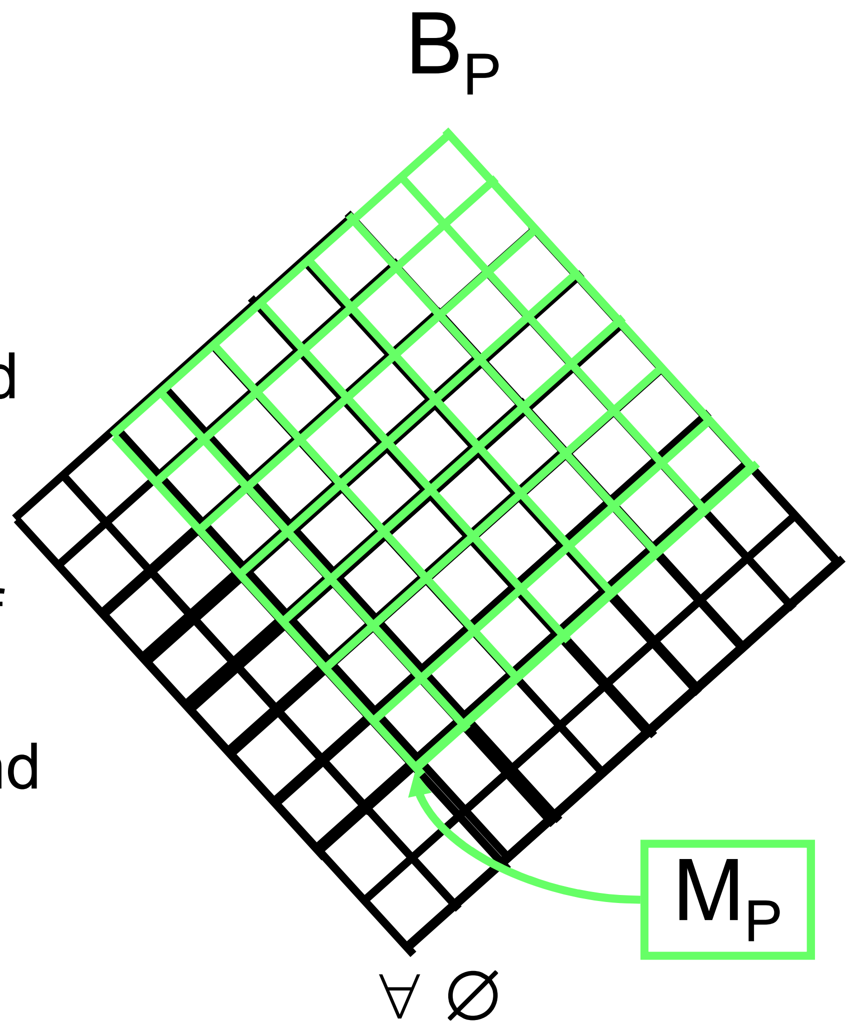
Let P be a definite program and $\{M_i\}_{i \in I}$ a non-empty set of Herbrand models for P. Then $\cap_{i \in I} M_i$ is an Herbrand model for P. If $\{M_i\}_{i \in I}$ contains all Herbrand models for p, then

$M_P := \cap_{i \in I} M_i$ is the *least Herbrand mode*l for P.

Idea

- $(2^{B_P}, \subseteq)$ is a lattice of all Herbrand interpretations of P with the bottom element $\emptyset$ and the top element B_P

- The least upper bound (lub) of a set of interpretations is the union, the greatest lower bound is the intersection.



Proposition:

Let P be a definite program and $\{M_i\}_{i \in I}$ a non-empty set of Herbrand models for P. Then $\cap_{i \in I} M_i$ is an Herbrand model for P.
If $\{M_i\}_{i \in I}$ contains all Herbrand models for p, then

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Proof:

$\cap_{i \in I} M_i$ is an Herbrand interpretation. Show, that it is a model.
Each definite program has B_P as model, hence I is not empty and one can show that M_P is a model.

Idea:

M_P is the „most natural“ model for P.

Theorem:

Let P be a definite program. Then

$M_P = \{A \in B_P : A$ is a logical consequence of $P\}$.

Proof:

A is a logical consequence of P iff

$P \cup \{\sim A\}$ is unsatisfiable iff

$P \cup \{\sim A\}$ has no Herbrand model iff

A is true wrt all Herbrand models of P iff

$A \in M_P$.

Definition

Let L be a lattice and $T:L\rightarrow L$ a mapping.
T is called *monotonic*, if $x\leq y$ implicates, that $T(x)\leq T(y)$.

Properties of Lattices

Definition

Let L be a lattice and $X \subseteq L$, X is called *directed*,
if each finite sub-set of X has an upper bound in X.

Definition

Let L be a lattice and $T: L \rightarrow L$ a mapping.
T is called *continuous*,if for each directed subset X
$T(lub(X))=lub(T(X))$.

Van Emden & Kowalski: The Semantics of Predicate Logic as a Programming Language, *J. ACM* 23, 4, 1976, pp. 733-742.

Definition

Let P be a definite program. The mapping $T_P: 2^{B_P} \rightarrow 2^{B_P}$ is defined as follows: Let I be an Herbrand interpretation. Then

$T_P(I)=\{ A \in B_P: A \leftarrow A_1 \wedge \ldots \wedge A_n$
is a ground instance of a clause in P and
$\{A_1,\ldots,A_n\} \subseteq I \}$

Practice

Let P be a definite program.

even(f(f(x)) $\leftarrow$ even(x).
odd(f(x) $\leftarrow$ even(x).
even(0).

Let $I_0=\emptyset$.

Then

$I_1=T_P(I_0)=?$

$T_P(I)=\{A\in B_P: A \leftarrow A_1 \wedge \ldots \wedge A_n$ is a ground instance of a clause in P and $\{A_1,\ldots,A_n\}\subseteq I\}$

Example

Let P be a definite program.

even(f(f(x)) ← even(x).
even(0).

Let $I_0=\emptyset$.

Then

$I_1=T_P(I_0)=\{even(0)\}$,

$I_2=T_P(I_1)=\{even(0), even(f(f(0))\}$,

$I_3=T_P(I_2)=\{even(0), even(f(f(0)),$
$even(f(f(f(f(0))))\},\ldots$

$T_P(I)=\{A\in B_P: A \leftarrow A_1 \wedge \ldots \wedge A_n$ is a ground instance of a clause in P and $\{A_1,\ldots,A_n\}\subseteq I\}$

T_P is monotonic.

Let P be a definite program.

plus(x,f(y),f(z)) <- plus(x,y,z)
plus(f(x),y,f(z)) <- plus (x,y,z)
plus(0,0,0)

Let $I_0=\emptyset$.

Then

$I_1=T_P(I_0)=?$

$T_P(I)=\{A\in B_P: A \leftarrow A_1 \wedge \ldots \wedge A_n$ is a ground instance of a clause in P and $\{A_1,\ldots,A_n\}\subseteq I\}$

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$T_P(I)=\{A \in B_P: A \leftarrow A_1 \wedge \ldots \wedge A_n$ is a ground instance of a clause in P and $\{A_1,\ldots,A_n\} \subseteq I\}$

Let $I_0=\emptyset$.

Then

$I_1=T_P(I_0)=\{plus(0,0,0)\}$

$I_2=T_P(I_1)=\{plus(0,0,0),$ $plus(f(0),0,f(0)),$ $plus(0,f(0),f(0))\}$

$I_3=T_P(I_2)=\{plus(0,0,0),$ $plus(f(0),0,f(0)),$ $plus(0,f(0),f(0)),$ $plus(f(f(0)),0,f(f(0))),$ $plus(0,f(f(0)),f(f(0))),$ $plus(f(0),f(0),f(f(0)),$ $plus(f(0),f(0),f(f(0)))\}\ldots$

Proposition

Let P be a definite program and I be an Herbrand interpretation of P. Then I is a model for P iff $T_P(I) \subseteq I$.

$\Rightarrow$ ✓

Let I be an Herbrand interpretation, which is not a model of P and $T_P(I) \subseteq I$. Then there exist ground instances $\{\sim A, A_1, \ldots, A_n\} \in I$ and a clause $A \leftarrow A_1 \wedge \ldots \wedge A_n$ in P. Then $A \in T_P(I) \subseteq I$. Refutation. ↯

Definition

let L be a complete lattice and T:L→L be monotonic. Then we define

$T\uparrow 0 = \bot$

$T\uparrow\alpha = T(T\uparrow(\alpha-1))$, if α is a successor ordinal

$T\uparrow\alpha = \text{lub}\{T\uparrow\beta: \beta<\alpha\}$ if α is a limit ordinal

$T\downarrow 0 = \top$

$T\downarrow\alpha = T(T\downarrow(\alpha-1))$, if α is a successor ordinal

$T\downarrow\alpha = \text{lub}\{T\downarrow\beta: \beta<\alpha\}$ if α is a limit ordinal

Example

Theorem

Let P be a definite program.
Then $M_P = lfp(T_P) = T_P\uparrow\omega$, where
$T\uparrow\alpha = T(T\uparrow(\alpha-1))$, if α is a successor ordinal
$T\uparrow\alpha = lub\{T\uparrow\beta : \beta<\alpha\}$, if α is a limit ordinal

Proof

$M_P = glb\{I : I \text{ is an Herbrand model for } P\}$
$= glb\{I : T_P(I) \subseteq I\}$
$= lfp(T_P)$ (not shown here)
$= T_P\uparrow\omega$.

Greatest Fixpoint not as easy to have:

- Example program
  - $p(f(x)) \leftarrow p(x)$
  - $q(a) \leftarrow p(x)$

- $T_P \downarrow \omega = \{q(a)\}$
- $T_P \downarrow (\omega+1) = \{\} = gfp(T_P)$

Definition

Let P be a definite program and G a definite goal. An *answer* for $P \cup \{G\}$ is a substitution for variables of G.

Definition

Let P be a definite program, G a definite goal $\leftarrow A_1 \wedge \ldots \wedge A_n$ and θ an answer for $P \cup \{G\}$.
We say that θ is a *correct answer* for $P \cup \{G\}$,
if $\forall((A_1 \wedge \ldots \wedge A_n)\theta)$ is a logical consequence of P.
The answer „no“ is *correct* if $P \cup \{G\}$ is satisfiable.

Example

Theorem

Let P be a definite program and G a definite goal $\leftarrow A_1 \wedge \ldots \wedge A_n$. Suppose θ is an answer for $P \cup \{G\}$, such that $(A_1 \wedge \ldots \wedge A_n)\theta$ is ground. Then the following are equivalent:

1. θ is correct
2. $(A_1 \wedge \ldots \wedge A_n)\theta$ is true wrt every Herbrand model of P
3. $(A_1 \wedge \ldots \wedge A_n)\theta$ is true wrt the least Herbrand model of P.

Proof

it suffices to show that 3 implies 1

$(A_1 \wedge \ldots \wedge A_n)\theta$ is true wrt the least Herbrand model of P implies

$(A_1 \wedge \ldots \wedge A_n)\theta$ is is true wrt all Herbrand models of P implies

$\sim(A_1 \wedge \ldots \wedge A_n)\theta$ is false wrt all Herbrand models of P implies

$P \cup \{\sim(A_1 \wedge \ldots \wedge A_n)\theta\}$ has no Herbrand models

implies $P \cup \{\sim(A_1 \wedge \ldots \wedge A_n)\theta\}$ has no models.