Procedural Semantics

Soundness of SLD-Resolution

Properties of Substitution

Propositions:

Let θ, ρ, γ be substitutions, E an expression.

$\theta\varepsilon=\varepsilon\theta=\theta$ (Identity)

$(E\theta)\rho = E(\theta\rho)$

$(\theta\rho)\gamma = \theta(\rho\gamma)$ (Associativity)

Proof:

Follows from definition of ε

prove proposition for E=x

prove $E(\theta\rho)\gamma = E\theta(\rho\gamma)$ for E=x and 2.

Example

Definition:

Let S be a finite set of simple expressions. A Substitution θ is called a *unifier* for S, if Sθ is a singleton.
A unifier θ is called a *most general unifier (mgu)* for S if, for each unifier ρ of S there exists a substitution γ such that $\rho=\theta\gamma$.

Example

Note: If there exist two mgu's then they are variants.

Disagreement Set

Definition:

Let S be a finite set of simple expressions. Locate the leftmost symbol position at which not all expressions in S have the same symbol and extract from each expression in S the subexpression beginning at that symbol position. The set of all such subexpressions is the *disagreement set*.

Example:

Let S={p(f(x),h(y),a), p(f(x),z,a), p(f(x),h(y),b)}, then the disagreement set is {h(y),z}

Unification Algorithm

1. put k:=0 and $\rho_0 := \varepsilon$

2. *If* Sρ_k is a singleton, *Then* *return*(ρ_k)
   Else *find the disagreement set* D_k of Sρ_k

3. *If* there exist a variable v and a term t in D_k such that v does not occur in t,
   // non-deterministic choice
   Then put $\rho_{k+1} := \rho_k\{v/t\}$, k++, *goto* 2
   Else exit // S is not unifiable

Example

$\rho_0 = \varepsilon$, k=1

$S\rho_0$={even(0),even(y)}

D_0={0,y}

choose variable y, term 0

put $\rho_1 := \varepsilon\{y/0\}$, k=1

$S\rho_1$={even(0)}

return.

Theorem:

Let S be a finite set of simple expressions. If S is unifiable, then the unification algorithm terminates and gives a mgu for s. If S is not unifiable, then the unification algorithm terminates and reports this fact.

Proof Sketch:

Assume θ is a unifier for S. Show that until termination for all k :

$\theta=\rho_k\gamma_k$

Unification Theorem

Proof Sketch:

Assume θ is a unifier for S. Show that until termination for all k : $\theta=\rho_k\gamma_k$

Induction start: $\rho_0=\varepsilon$, $\gamma_0 = \theta$

From k to k+1 (we only need to consider $S\rho_k$, because otherwise, we are done):

$|S\theta| = 1 \Rightarrow |D\gamma_k| = 1$

Pick a variable v and a term t, then:

- $v\gamma_k = t\gamma_k$
- $\rho_{k+1} = \rho_k\{v/t\}$
- $\gamma_{k+1} = \gamma_k \setminus \{v/v\gamma_k\}$, i.e. if v is bound in γ_k then
  - $\gamma_k = \{v/v\gamma_k\} \cup \gamma_{k+1} = \{v/t\gamma_k\} \cup \gamma_{k+1} = \{v/t\gamma_{k+1}\} \cup \gamma_{k+1} = \{v/t\}\gamma_{k+1}$
  - $\theta=\rho_k\gamma_k = \rho_k\{v/t\}\gamma_{k+1} = \rho_{k+1}\gamma_{k+1}$

SLD-Resolution

- SLD: SL-resolution for definite clauses
- SL: Linear resolution with selection function

Definition:

Let G be $\leftarrow A_1,\ldots., A_m,\ldots, A_k$

and C be $A \leftarrow B_1,\ldots., B_q$.

Then G' is *derived* from G and C using mgu θ, if:

a. A_m is an Atom, called the *selected* atom, in G

b. θ is an mgu of A_m and A.

c. G‘ is the goal $\leftarrow (A_1,\ldots., B_1,\ldots., B_q,\ldots, A_k)\theta$.

In resolution terminology G‘ is called a *resolvent* of G and C.

Definition:

Let P be a definite program and G_0 a definite goal.
An *SLD-Derivation* of $P \cup \{G_0\}$ consists of a (finite or infinite) sequence $G_0, G_1, G_2, \ldots$ of goals, a sequence $C_1, C_2, \ldots$ of variants of program clauses of P and a sequence

$\theta_1, \theta_2, \ldots$ of mgu's such that each G_{i+1} is derived from G_i and C_{i+1} using θ_{i+1}.

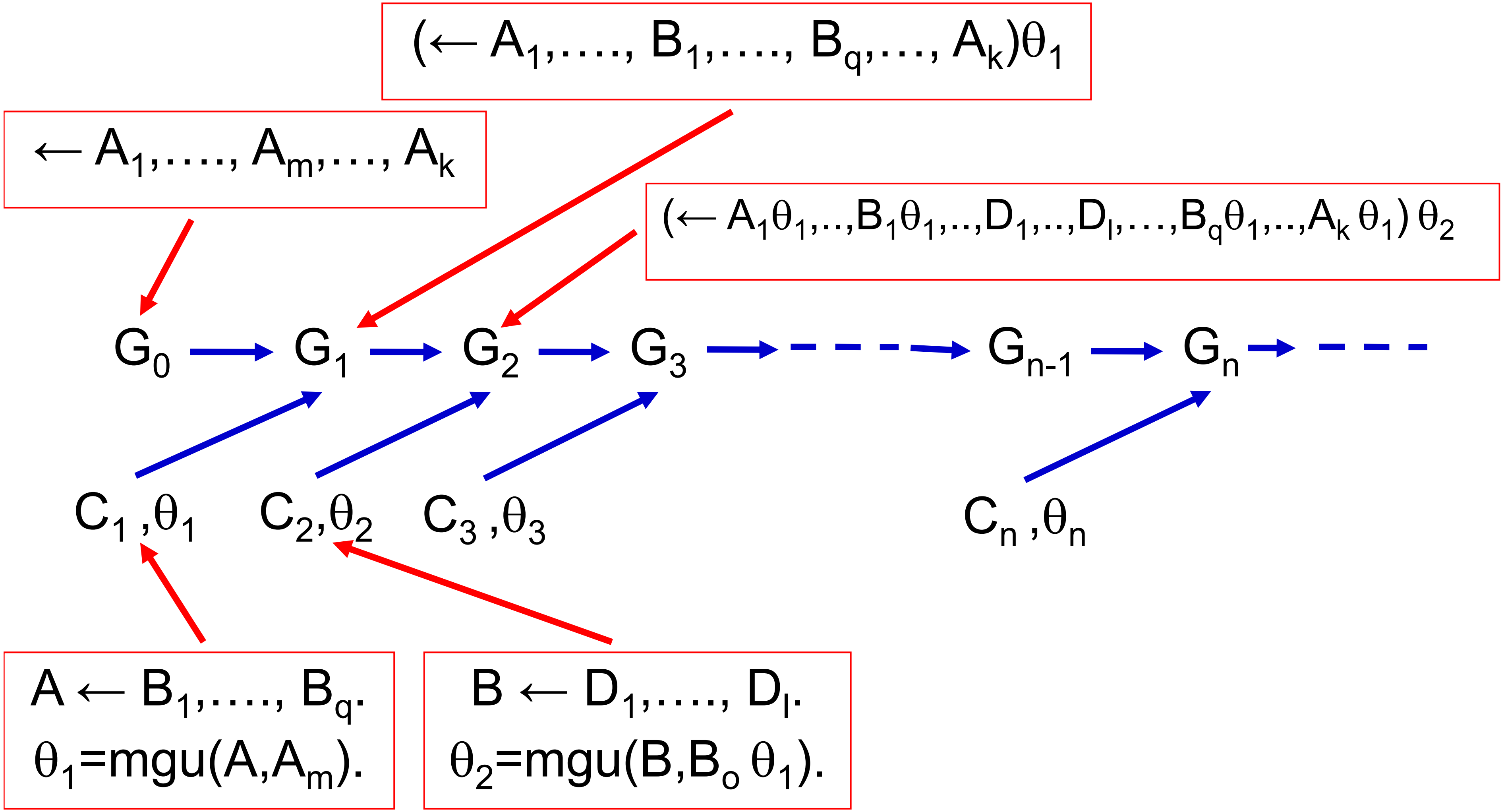
standardising apart the variables:

subscribe all variables in C_i with i.

Otherwise $\leftarrow p(x)$. could not be unified with $p(f(x)) \leftarrow$.

each program clause variant $C_1, C_2, \ldots$ is called an *input clause* of the derivation

SLD-Derivation visualised



Example – Restricted SLD-Refutation

Program P
1 Q(x) :- R(g(x)).
2 R(y).

Goal: Q(f(z)).

$\leftarrow$ Q(f(z))

$\leftarrow$ R(g(f(z))).

$G_0 \rightarrow G_1 \rightarrow \square$

C_1, θ_1 $\quad$ C_2, θ_2

Q(x) :- R(g(x)).
θ_1=mgu(Q(x),Q(f(z)))
={x/f(z)}

R(y)$\leftarrow$.
θ_2=mgu(R(y), R(g(f(z))))
= {y/g(f(z))}

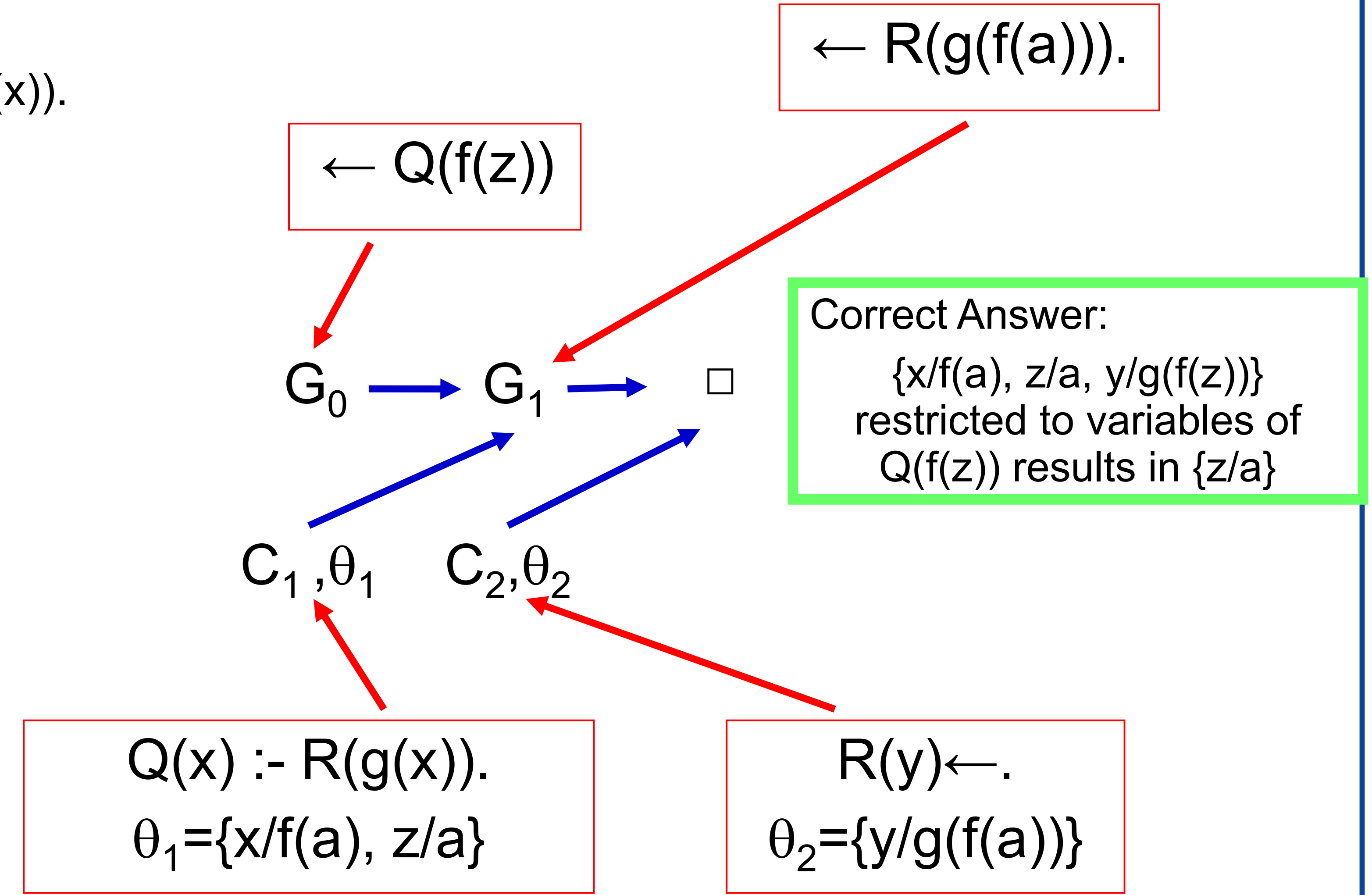
Computed Answer
{x/f(z), y/g(f(z))} restricted to variables of Q(f(z)) results in
ε

Example – Unrestricted SLD-Refutation

unrestricted → unifiers need not be mgu's

Program P
1 Q(x) :- R(g(x)).
2 R(y).

Goal: Q(f(z)).



Definition:

An *SLD-refutation* of P ∪ {G} is a finite SLD-derivation of P ∪ {G}, which has □ as the last goal in the derivation. If G_n=□, we say the refutation has *length n*.

SLD-derivations can be *finite* or *infinite*.

A finite SLD-derivation can be *successful* or *fail*.

An SLD-derivation is successful, if it ends in □**.**

An SLD-derivation is *failed*, if it ends in a non-empty goal, which cannot be unified with the head of a program clause.

Definition:

Let P be a definite program.
The *success set* of P is the set of all $A \in B_P$ such that $P \cup \{\leftarrow A\}$ has an SLD-refutation.

Procedural Counterpart of the Least Herbrand Model!

Definition:

Let P be a definite program and G a definite goal. Let $\theta_1 \ldots \theta_n$ be the sequence of mgu's used in an SLD-refutation of $P \cup \{G\}$.

A *computed answer* θ for $P \cup \{G\}$ is the substitution obtained by restricting the composition $\theta_1 \ldots \theta_n$ to the variables of G.

Example: P=Slowsort

goal:

$\leftarrow$ sort(17.22.6.5.nil,y)

computed answer:

{y/5.6.17.22.nil}

sort(x,y) $\leftarrow$ sorted(y), perm(x,y)
sorted(nil) $\leftarrow$
sorted(x.nil) $\leftarrow$
sorted(x.y.z) $\leftarrow$ x$\leq$y, sorted(y.z)
perm(nil,nil) $\leftarrow$
perm(x.y,u.v) $\leftarrow$ delete(u,x.y,z),perm(z,v)
delete(x,x.y,y) $\leftarrow$
delete(x,y.z,y.w) $\leftarrow$ delete(x,z,w)
0$\leq$x $\leftarrow$
f(x) $\leq$f(y) $\leftarrow$ x$\leq$y.

Theorem

Let P be a definite program and G a definite goal. Then every computed answer for $P \cup \{G\}$ is a correct answer for $P \cup \{G\}$.

Proof

Let G be the goal $\leftarrow A_1,\ldots, A_k$ and $\theta_1\ldots\theta_n$ the sequence of mgu's in a refutation of $P \cup \{G\}$.
Show that $\forall((A_1,\ldots, A_k)\theta_1\ldots\theta_n)$is a logical consequence of P using induction (starting at the last goal) over the length of the derivation.

Corollary

The success set of a definite program is contained in its least Herbrand model.

Proof

Let the program be P, let $A \in B_P$ and suppose $P \cup \{\leftarrow A\}$ has a refutation. By the theorem on the prior slide A is a logical consequence of P. Thus A is in the least Herbrand model of P.

- **strengthen this corollary**
  If $A \in B_P$ and $P \cup \{\leftarrow A\}$ has a refutation of length n, then $A \in T_P \uparrow n$.

- **Notation**
  $[A]=\{A' \in B_P: A'=A\theta$ for some substitution $\theta\}$

Completeness of SLD-Resolution

- Not treated this year, check out

- http://isweb.uni-koblenz.de/Teaching/SS08/adm08