Models in First Order Logics
Overview

First-order logic. Syntax and semantics.
Herbrand interpretations;
Clauses and goals;
Datalog.
First-order signature « consists of

- \( \text{con} \) — the set of constants of «;
- \( \text{fun} \) — the set of function symbols of «;
- \( \text{rel} \) — the set of relation symbols of «.
Terms

Term of « with variables in $X$:
1. Constant $c \in con$;
2. Variable $v \in X$;
3. If $f \in fun$ is a function symbol of arity $n$ and $t_1, \ldots, t_n$ are terms, then $f(t_1, \ldots, t_n)$ is a term.

A term is ground if it has no variables $\text{var}(t)$ — the set of variables of $t$
Abstract and concrete notation

Abstract notation:
- $a, b, c, d, e$ for constants;
- $x, y, z, u, v, w$ for variables;
- $f, g, h$ for function symbols;
- $p, q$ for relation symbols,

Example: $f(x, g(y))$.

Concrete notation:
- Variable names start with upper-case letters.

Example: likes(john, Anybody).
Atomic formulas, or atoms $p(t_1, \ldots, t_n)$.

$(A_1 - \ldots - A_n)$ and $(A_1 \ \lor \ldots \ \lor A_n)$

$(A \ \triangleleft \ B)$ and $(A \ \leftrightarrow \ B)$

$= A$

$\lor A$ and $<vA$
Substitutions

Substitution \( \theta \) is any mapping from the set \( V \) of variables to the set of terms such that there is only a finite number of variables \( v \in V \) with \( \theta(v) \neq v \).

Domain \( \text{dom}(\theta) \), range \( \text{ran}(\theta) \) and variable range \( \text{vran}(\theta) \):

\[
\text{dom}(\theta) = \{ v | v \neq \theta(v) \},
\]

\[
\text{ran}(\theta) = \{ t | \exists v \in \text{dom}(\theta) (\theta(v) = t) \},
\]

\[
\text{vran}(\theta) = \text{var(} \text{ran}(\theta) \text{)}.
\]

Notation: \( \{ x_1 \mapsto t_1, \ldots , x_n \mapsto t_n \} \)

empty substitution \( \{ \} \)
Application of a substitution \( \theta \) to a term \( t \):

- \( x \theta = \theta(x) \)
- \( c \theta = c \)
- \( f(t_1, \ldots, t_n) \theta = f(t_1 \theta, \ldots, t_n \theta) \)
A Herbrand interpretation of a signature \( \Sigma \) is a set of ground atoms of this signature.
Truth in Herbrand Interpretations

1. If A is atomic, then \( I \models A \) if \( A \in I \)

2. \( I \models B_1 \land \ldots \land B_n \) if \( I \models B_i \) for all \( i \)

3. \( I \models B_1 \lor \ldots \lor B_n \) if \( I \models B_i \) for some \( i \)

4. \( I \models B_1 \rightarrow B_2 \) if either \( I \models B_2 \) or \( I \not\models B_1 \)

5. \( I \models \neg B \) if \( I \not\models B \)

6. \( I \models \forall x B \) if \( I \models B\{x \mapsto t\} \) for all ground terms \( t \) of the signature

7. \( I \models \exists x B \) if \( I \models B\{x \mapsto t\} \) for some ground term \( t \) of the signature
A formula $F$ is a tautology (is valid), if $I \models F$ for every (Herbrand) interpretation $I$.

A formula $F$ is inconsistent, if $I \not\models F$ for every (Herbrand) interpretation $I$. 
Types of Formulas

- Tautologies / Valid
- Neither valid nor inconsistent
- Inconsistent

All formulas

Invalid formulas
Logical Implication

A (set of) formula(s) $F$ logically implies $G$ (we write $F \models G$), iff every (Herbrand) interpretation $I$ that fulfills $F$ ($I \models F$) also fulfills $G$ ($I \models G$).

$\neg F \models G$ is true iff for every (Herbrand) interpretation $I$:
$I \models (\neg F \not\models G)$
Literals

Literal is either an atom or the negation \( \neg A \) of an atom \( A \).

Positive literal: atom

Negative literal: negation of an atom

Complementary literals: \( A \) and \( \neg A \)

Notation: L
Clause: (or normal clause) formula $L_1 - ... - L_n \rightarrow A$, where $n \geq 0$, each $L_i$ is a literal and $A$ is an atom.

Notation: $A :- L_1 - ... - L_n$ or $A :- L_1, ... , L$

Head: the atom $A$.

Body: The conjunction $L_1 - ... - L_n$

Definite clause: all $L_i$ are positive

Fact: clause with empty body
### Syntactic Classification

<table>
<thead>
<tr>
<th>Clause</th>
<th>Class</th>
</tr>
</thead>
<tbody>
<tr>
<td>lives(Person, sweden) :- sells(Person, wine, Shop), not open(Shop, saturday)</td>
<td>normal</td>
</tr>
<tr>
<td>spy(Person) :- russian(Person)</td>
<td>definite</td>
</tr>
<tr>
<td>spy(bond)</td>
<td>fact</td>
</tr>
</tbody>
</table>
Goal

- Goal (also normal goal) is any conjunction of literals $L_1 \land \ldots \land L_n$
- Definite goal: all $L_i$ are positive
- Empty goal $\Box$: when $n = 0$
Excercise:
- Syntax & Semantics for Monadic Fuzzy Logics

- Monadic: only unary predicates
- Fuzzy:
  - Truth values in $[0,1]$
  - Operators for truth values
    - $T$-norm: $\land_{\text{min}}(a,b) = \min\{a,b\}$
    - $T$-conorm: $\lor_{\text{min}}(a,b) = \max\{a,b\}$
    - Also: $\land_{\text{prod}}(a,b) = a \cdot b$, $\lor_{\text{sum}}(a,b) = a + b - a \cdot b$, ...
    - Duality for $N(x) = 1 - x$