Advanced Data Modeling

Minimal Models

Steffen Staab

People and Knowledge Networks

Overview

- Logic as query language.
- Grounding.
- Minimal Herbrand models.
- Completion.

Given:

- first-order formula $A[x_1, \ldots, x_n]$
- Herbrand interpretation I

- This first-order formula can be considered as a definition of a relation R_A on $T^{n}{}_{§}$ as follows:

    - $(t_1, \ldots, t_n)$ 2 R_A := I ² $A[t_1, \ldots, t_n]$

Clauses as definitions

We say that a clause

$p(t_1, \ldots , t_m) \text{ :- } L_1 , \ldots , L_n$

defines the relation symbol *p*.

Let C be a set of clauses and *p* be a relation symbol. We call the definition of *p* in C the set of all clauses in C that define *p*.

Principles of semantics

A deductive database is a **set of clauses.**

This set of clauses is regarded as a **collection of definitions** of relations.

The **semantics** defines the meaning of this definitions by associating with them an **interpretation**, or a class of interpretations.

Query answering is based on the semantics.

Two key assumptions

- the **unique name assumption**:
  each name denotes a unique object.

- the **closed world assumption**:
  - a negative statement :*A* holds if the corresponding positive one *A* does not hold.

Both assumptions are not supported by (Tarskian) models for first-order logics (**why not?**) → One solution: minimal models

Minimal Herbrand Models

Let *I* be a Herbrand model of a set of formulas *S*.

We call *I* a minimal Herbrand model of *S* if it is minimal w.r.t. the subset relation, i.e. for every Herbrand model *I´* of *S* of the same signature we have *I´* ¶ *I*.

I is called the least Herbrand model of *S* if for every Herbrand model *I´* of *S* of the same signature we have *I* µ *I´*.

Does every set of formulas S have a least Herbrand model?

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Counterexample for normal clauses:

person(a).

man(X) :- person(X), not woman(X).

woman(X) :- person(X), not man(X).

Does every set of formulas S have a least Herbrand model?

What about definite clauses?

Let E, E´ be a pair of terms or formulas.

E´ is an **instance** of E, denoted E > E´, if there exists a substitution μ such that Eμ = E´.

ground instance: instance that is ground,

E´ is a **variant** of E if E´ is an instance of E and E is an instance of E´.

- P(x,a) is instance of P(x,y) because of P(x,y)[y|a]

- P(b,a) is a ground instance

- P(x,y) and P(u,v) are variants of each other, because of
  - [x|u, y|v] and
  - [u|x, v|y]

Grounding

- Let C be a set of clauses and § be any signature containing all symbols used in C. The **grounding of C w.r.t. §,** denoted **C*** is the set of all ground instances of the signature § of clauses in C.

- Lemma. Let I be a Herbrand interpretation and C be a set of clauses. Then I ² C if and only if I ² C*.

General proof scheme

$$
\begin{tikzcd}
\text{First order formula} \arrow[r, "\text{calculus}"] \arrow[d, "\text{(all possible ways of) grounding}"'] & \text{First order Consequences} \\
\text{Propositional formula} \arrow[r, "\text{calculus}"] & \text{Propositional Consequences} \arrow[u, "\text{lifting}"]
\end{tikzcd}
$$

- Proof

Logic with equality

- Additional atomic formulas **s = t**, where s, t are terms.

- Abbreviation: $x \neq y$:= :(x = y).

- Unlike other relations, the semantics of s = t is **predefined** in all Herbrand interpretations:
  I ² s = t if s coincides with t.

Example valid formulas

- $f(x_1, \ldots , x_n) = f(y_1, \ldots y_n)\ !\ x_1 = y_1\ \text{Æ} \ldots \text{Æ}\ x_n = y_n$
- $f(x_1, \ldots , x_n) \neq g(y_1, \ldots , y_n)$
- $f(x_1, \ldots ,x_n) \neq c$
- $d \neq c$
- $A[t]\ \$\ 8x(x = t\ !\ A[x])$

Semantics of definitions

Consider a definition of a relation *r*

$$r(\bar{t}_1) :- G_1$$

$$\ldots$$

$$r(\bar{t}_m) :- G_m$$

What is the meaning of this definition?

Is there a largest Herbrand model?

Especially in the light of negation as failure?

Idea of completion

man(hans).

man(adam).

person(eva).

woman(X) :- person(X), not man(X).

Is eva a woman?

> She might be a man and we just don‘t know!
>
> Minimal model says that she is not a woman!

Trick: Completion:

man(X) $ (X=hans Ç X=adam).

woman(X) $ X=eva.

I.e. hans and adam are the only men.

Idea of Completion

man(hans).

man(X) :- lovesBeer(X,Y).

Completion:

man(X) $ X=hans Ç (9 Y: X=U Æ lovesBeer(U,Y))

A man is a man only if he is hans or if he loves some brand of beer.

Completion. Step 1.

Replace every clause by an equivalent one such that the arguments of r are $x_1, \ldots, x_n$:

Given:

$r(t_1, \ldots, t_n)$:- G

Replace by:

$r(x_1, \ldots, x_n)$:- $x_1 = t_1$ Æ … Æ $x_n = t_n$ Æ G

Completion. Step 2.

If there are variables y_1, … , y_k occurring in a body but not in the head, apply 9 to these variables, i.e.,

Given

$r(x_1, \ldots , x_n)$:- G

Modify to

$r(x_1, \ldots , x_n)$:- 9 y_1 … 9 y_k G

Completion. Step 3.

If there are several definitions, replace them by one

Given

$r(x_1, \ldots ,x_n)$:- G_1

…

$r(x_1, \ldots , x_n)$:- G_m

Replace by

$r(x_1, \ldots , x_n)$:- G_1 Ç … Ç G_m

Completion. Step 4.

Replace :- by $:

Given

$r(x_1, \ldots, x_n)$:- G_1Ç … Ç G_m

Replace by

$r(x_1, \ldots, x_n)$ **$** G_1Ç … Ç G_m

The formula

$r(x_1, \ldots, x_n)$ **$** G_1Ç … Ç G_m

is called the **completed definition** of the original set of clauses.

Example

r(u,v):-p(u,1,z).

r(v,u):-p(2,u,z).

r(x,y) :- x=u Æ y=v Æ p(u,1,z).

r(x,y) :- x=v Æ y=u Æ p(2,v,z).

r(x,y) :- 9z,u,v x=u Æ y=v Æ p(u,1,z).

r(x,y) :- 9z,u,v x=u Æ y=v Æ p(2,v,z).

r(x,y) $ (9z,u,v x=u Æ y=v Æ p(u,1,z)) Ç (9z,u,v x=u Æ y=v Æ p(2,v,z))

Properties

- All steps **preserve Herbrand models**, except for the last one.
  - Why?

- Gives a **unique semantics to non-recursive definitions**
  - What about recursive definitions?

- Logic programming **semantics and first-order semantics coincide for definite programs**

Recursive definitions

odd(1).

even(f(X)) :- odd(X).

odd(f(X)) :- even(X).

Completion:

odd(X) $ X=1 Ç (X=f(Y) Æ even(Y)).

even(X) $ X=f(Y) Æ odd(Y).

Recursive definitions

person(adam).
person(eva).
woman(X) :- person(X), not man(X).
man(X) :- person(X), not woman(X).

Completion:
woman(X) $ (Y=X Æ person(Y) Æ not man(Y)).
man(X) $ (Y=X Æ person(Y) Æ not woman(Y)).

Semantics not unique in logic programming:
Models are
I={woman(adam),woman(eva)}
I={man(adam),man(eva)}
I={woman(adam),man(eva)}
I={man(adam),woman(eva)}

What is the semantics in first order logics?

Recursive definitions

person(adam).
person(eva).
woman(X) :- person(X), not man(X).
man(X) :- person(X), not woman(X).

Completion:
woman(X) $ (Y=X Æ person(Y) Æ not man(Y)).
man(X) $ (Y=X Æ person(Y) Æ not woman(Y)).

Semantics not unique in logic programming:
Models are (add {person(adam),person(eva)} to each)
I={woman(adam),woman(eva)}
I={man(adam),man(eva)}
I={woman(adam),man(eva)}
I={man(adam),woman(eva)}
What is the semantics in first order logics?
Models are: $woman^I=\{\}=man^I$

Simple characterization of completion

Let C be a definition of r, I be a Herbrand model of the corresponding completed definition, and $r(t_1, \ldots, t_n)$ be a ground atom.

Then

I ² $r(t_1, \ldots, t_n)$,

9 $(r(t_1, \ldots, t_n)$:- $L_1, \ldots, L_m)$ 2 C^* (I ² L_1 Æ … Æ L_n):

Immediate consequence operator

$T_C(I) :=$ { A | there exists (A :- G) 2 C^*
such that $I\ {}^{2}\ G$ }

Fixpoint: an interpretation such that $T_C(I) = I$.

Definite clauses have the least model

Let C be a set of definite clauses.

Define

I_0 := {}

I_{n+1} := $T_C(I_n)$, for all n ¸ 0,

I! := [$^!_{i=0}$ I_i

Then $I_!$ is the least fixpoint of T_C and also the least Herbrand model of C.

Let C be a non-recursive database and K be an arbitrary interpretation.

Define

I_0 := **K**

$I_{n+1} := T_C(I_n)$, for all n ¸ 0,

I! := $[^!_{i=0}$ I_i

Then $I_!$ is the only fixpoint of T_C. Moreover, for some n we have $I_! = I_n$.

Non-recursive sets of clauses

- Let C be a set of clauses.

- Its **dependency graph** consists of pairs *p ! r* such that *p* occurs in the body of a clause which defines *r* in C.

- A set of clauses is **non-recursive** if the dependency graph contains no cycles.

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Its **dependency graph** consists of pairs *p ! r* such that *p* occurs in the body of a clause which defines *r* in C.

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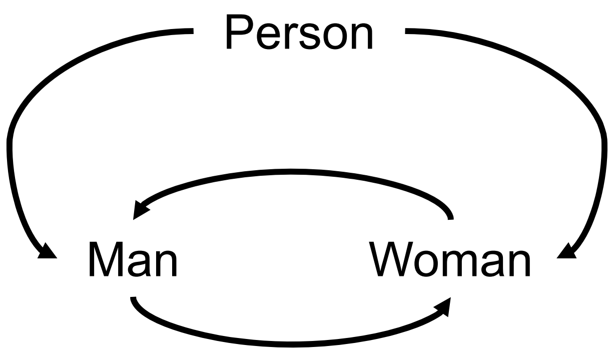
Person(a).

Woman(X) :- Person(X), Not Man(X).

Man(X) :- Person(X), Not Woman(X).



Non-recursive sets of clauses

Dependency graph of C consists of pairs p ! r such that p occurs in the body of a clause which defines r in C.

C is non-recursive if the dependency graph contains no cycles.

A relation p depends on a relation q in C if there exists a path of length ¸ 1 from q to p in the dependency graph of C.

A set of clauses is non-recursive if and only if no relation depends on itself.