Stable Models

See Bry et al 2007
Controversy

p :- not p.

Justification postulate

- Requests dependable justifications for derived truths.

⇒ Some programs do not have a model (cf above)

Consistency postulate

- Every syntactically correct set of normal clauses is consistent and must therefore have a model.

⇒ \{p\} is a model for the program above
Controversy

\[ p : \neg q \]
\[ q : \neg p \]

Justification postulate

- Requests dependable justifications for derived truths.

⇒ Both \( \{p\} \) and \( \{q\} \) are reasonable models.

Consistency postulate

- Every syntactically correct set of normal clauses is consistent and must therefore have a model.
Definition: Gelfond-Lifschitz transformation

Let $S$ be a (possibly infinite) set of ground normal clauses, i.e., of formulas of the form

$$A :- L_1, \ldots, L_n$$

where $n \geq 0$ and $A$ is a ground atom and the $L_i$ are ground literals. Let $B \subseteq B_P$.

The Gelfond-Lifschitz transform $GL_B(S)$ of $S$ with respect to $B$ is obtained from $S$ as follows:

1. remove each clause whose antecedent contains a literal $\neg A$ with $A \in B$.
2. remove from the antecedents of the remaining clauses all negative literals.
Example Gelfond-Lifschitz transformation

Program:

\[
\text{brother}(X,Y) :- \text{brother}(X,Z), \text{brother}(Z,Y), \text{not } = (X,Y).
\]
\[
\text{brother}(\text{chico},\text{harpo}).
\]
\[
\text{brother}(\text{harpo},\text{chico}).
\]

Grounded Program:

\[
\text{brother}(\text{chico},\text{chico}) :- \text{brother}(\text{chico},\text{harpo}), \text{brother}(\text{harpo},\text{chico}), \text{not } = (\text{chico},\text{chico})
\]
\[
\text{brother}(\text{chico},\text{harpo}) :- \text{brother}(\text{chico},\text{chico}), \text{brother}(\text{chico},\text{harpo}), \text{not } = (\text{chico},\text{harpo})
\]
\[
\ldots [5 \text{ more}] \ldots
\]
\[
\text{brother}(\text{harpo},\text{harpo}) :- \text{brother}(\text{harpo},\text{chico}), \text{brother}(\text{chico},\text{harpo}), \text{not } = (\text{harpo},\text{harpo})
\]
\[
\text{brother}(\text{chico},\text{harpo}).
\]
\[
\text{brother}(\text{harpo},\text{chico}).
\]
Example Gelfond-Lifschitz transformation

S={
brother(chico,chico) :- brother(chico,harpo),brother(harpo,chico), not =(chico,chico)
brother(chico,harpo) :- brother(chico,chico),brother(chico,harpo), not =(chico,harpo)
…[5 more]…
brother(harpo,harpo) :- brother(harpo,chico), brother(chico,harpo), not =(hharpo,harpo)
brother(chico,harpo).
brother(harpo,chico).
}
Ex 1: B={brother(chico,harpo), brother(harpo,chico), =(chico,chico), =(harpo,harpo)}
GL_B(S)=

brother(chico,chico) :- brother(chico,harpo),brother(harpo,chico), not =(chico,chico)
brother(chico,harpo) :- brother(chico,chico),brother(chico,harpo), not =(chico,harpo)
…[5 more]…
brother(harpo,harpo) :- brother(harpo,chico), brother(chico,harpo), not =(hharpo,harpo)
brother(chico,harpo), brother(harpo,chico). }
Example Gelfond-Lifschitz transformation

\[ S = \{
\text{brother(chico, chico)} :- \text{brother(chico, harpo), brother(harpo, chico), not } = (\text{chico, chico})
\]

\[ \text{brother(chico, harpo)} :- \text{brother(chico, chico), brother(chico, harpo), not } = (\text{chico, harpo}) \]

…[5 more]…

\[ \text{brother(harpo, harpo)} :- \text{brother(harpo, chico), brother(chico, harpo), not } = (\text{harpo, harpo}) \]

\[ \text{brother(chico, harpo)}.
\]

\[ \text{brother(harpo, chico)}. \}

\]

Ex 1: \( B = \{ \text{brother(chico, harpo), brother(harpo, chico), } = (\text{chico, chico}), = (\text{harpo, harpo}) \} \)

\[ \text{GL_B(S)} = \{
\text{brother(chico, chico)} :- \text{brother(chico, harpo), brother(harpo, chico), not } = (\text{chico, chico})
\]

\[ \text{brother(chico, harpo)} :- \text{brother(chico, chico), brother(chico, harpo), not } = (\text{chico, harpo}) \]

…[5 more]…

\[ \text{brother(harpo, harpo)} :- \text{brother(harpo, chico), brother(chico, harpo), not } = (\text{harpo, harpo}) \]

\[ \text{brother(chico, harpo)}, \text{ brother(harpo, chico)}. \} \]
Stable model semantics

**Definition** (stable model):
Let $S$ be a (possibly infinite) set of ground normal clauses. An Herbrand interpretation $B$ is a stable model of $S$, iff it is the unique minimal Herbrand model of $GL_B(S)$.

**Note:**
A stable model of a set $S$ of normal clauses is a stable model of the (possibly infinite) set of ground instances of $S$.

**Lemma:** Let $S$ be a set of ground normal clauses and $B$ an Herbrand interpretation. $B \models S$ iff $B \models GL_B(S)$
Stable model semantics

**Definition** (stable model):
Let $S$ be a (possibly infinite) set of ground normal clauses. An Herbrand interpretation $B$ is a stable model of $S$, **iff** it is the unique minimal Herbrand model of $GL_B(S)$.

**Note:**
A stable model of a set $S$ of normal clauses is a stable model of the (possibly infinite) set of ground instances of $S$.

**Lemma:** Let $S$ be a set of normal clauses. Each stable model of $S$ is a minimal Herbrand model of $S$. 
Examples

\[ S_1 = \{ \]
\[ \quad (\ p :\not p \ ), \]
\[ \quad (\ p :\text{true} \ ) \} \]

Has the stable model \{p\}.

\[ \text{GL}_{\{p\}}(S) = \{(p:\text{true})\}, \text{which has the unique minimal model } \{p\} \]

It has no other model.
Examples

\[ S_2 = \{ \ ( p :\neg p ) \} \]

has no stable model.

It has the model \( \{ p \} \), but \( \text{GL}_{\{p\}}(S) = \{ \} \), which has the unique minimal model \( \{ \} \).

It has the model \( \{ \} \), but \( \text{GL}_{\{p\}}(S) = \{ ( p:\text{true} ) \} \), which has the unique minimal model \( \{ p \} \).
$S_3 = \{
    (q \iff r, \neg p),
    (r \iff s, \neg t),
    (s \iff \text{true})\}

Has the following models:
\{s,r,q\}, \{s,t,q\}, \{s,t,p\}, \ldots

But after applying $GL_B(S)$ $p$ and $t$ cannot be part of the unique minimal model and $\{s,r,q\}$ must be!

Therefore it has the single stable model $\{s,r,q\}$
Examples

\[ S_4 = \{ \]
\[ ( q : - \text{not } p ), \]
\[ ( p : - \text{not } q ) \} \]

Has the following models:
- \{q\}, \{p\}

Both are stable models!
Cautious vs brave (skeptical vs credulous)

Logical consequence in stable model semantics

- Cautious (skeptical) entailment:
  - $P \models F$, iff $F$ is true in all stable models of $P$

- Brave (credulous) entailment:
  - $P \models F$, iff $F$ is true in some stable model of $P$

- Main interest typically:
  - The different models with their different properties
Observations on stable models

• Stable model semantics coincides with the intuitive understanding based on the „justification postulate“.

• Unintuitive minimal models of the examples turn out not to be stable and the stability criterion retains only those minimal modes that are intuitive.

• A set may have several stable models or exactly one or none

• Each stratifiable set has exactly one stable model.
Example well-founded model

\[ S_1 = \{ (p : \neg p), (p : \text{true}) \} \]

Has the well-founded model (\{p\},\{\})
Example well-founded model

\[ S_2 = \{ (p \leftarrow \neg p) \} \]

has the well founded model (\{\},\{\})
Examples

\[ S_3 = \{ \]
\[ \quad ( q :- r, \text{not} \ p ), \]
\[ \quad ( r :- s, \text{not} \ t ), \]
\[ \quad ( s :- \text{true} ) \} \]

Has the well-founded model \((\{s,r,q\},\{t,p\})\)
Examples

$S_4 = \{ 
\begin{align*}
& q :- \neg p, \\
& p :- \neg q
\end{align*}
\}\}

Has the well-founded model (\{\},\{\})
Comparison

Stable model semantics

- Justification postulate

Well-founded semantics

- Always one model / consistency postulate

If a rule set is stratifiable, then it has a unique minimal model, which is a stable model and at the same time a total well-founded model.

If a rule set $S$ has a total well-founded model, then this model is also the single stable model of $S$ and vice versa.

If a rule set $S$ has a partial well-founded model $I$ that is not total, then $S$ has either no stable model or more than one. In this case a ground atom is true (or false, respectively) in all stable models of $S$ if and only if it is true in $I$ (or false, respectively).
Stable model semantics

• Justification postulate

Well-founded semantics

• Always one model / consistency postulate

Well-founded semantics convey the „agreement“ of stable models.

Well-founded semantics cannot distinguish between several justifiable models ($S_4$) and no justifiable model ($S_2$)
Side remark

\[ p :- \text{odd}(X), \neg \text{odd}(X). \]
\[ \text{odd}(s(X)) :- \neg \text{odd}(X). \]

Well founded model is:
\[ I_{2n} = ( \{\text{odd}(s(0)), \text{odd}(s(s(s(0))))), \ldots, \text{odd}(s^{2n-1}(0))\}, \]
\[ \{\text{odd}(0), \text{odd}(s(s(0))), \ldots, \text{odd}(s^{2n-2}(0))\}) \]

Fixpoint: \[ I_{\omega+1} = I_{\omega} \cup (\{\},\{p\}) \]
d.h. \( \neg p \)

WFS is undecidable, NAF is semi-decidable