Network Theory and Dynamic Systems

Game Theory: Mixed Strategies

Prof. Dr. Steffen Staab
- Rationale to randomize
  - Matching pennies do not result in a Nash equilibrium, because opponent would switch if he knew your choice

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→ Randomize your choice such that your opponent cannot determine the best response
→ Mixed strategy
**Nash Equilibrium**

**Definition:** A Nash equilibrium for a mixed-strategy game is a pair of strategies \((S, T)\), where \(S\) and \(T\) are both probability distributions over individual strategies available to player 1 and 2, respectively, such that each is a best response to the other.

Can a pure strategy lead to a Nash equilibrium for the pennies game?
Mixed Strategy

- Each player chooses a probability for each strategy
  - Player 1 chooses Head with $p$ and Tail with $1-p$
  - Player 2 chooses Head with $q$ and Tail with $1-q$

- Reward for player 1:
  - $pq \cdot 1 + (1-p)(1-q) \cdot 1 + p(1-q) \cdot (-1) + (1-p)q \cdot (-1) = pq + 1 + pq - p - q - p + pq - q + pq = 1 - 2(p+q) + 4pq$

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Mixed Strategy

- Reward for player 1:
  - \( pq \cdot 1 + (1-p)(1-q) \cdot 1 + p(1-q) \cdot (-1) + (1-p)q \cdot (-1) = \)
  - \( pq + 1 + pq - p - q - p + pq - q + pq = \)
  - \( 1 - 2(p+q) + 4pq \)

- Reward for player 2: Multiply with -1: \(-1 + 2(p+q) - 4pq\)

- Choose optimum: differentiate for both variables
  - \( d(1 - 2(p+q) + 4pq)/dp = -2 + 4q = 0 \)
  - \( d(1 - 2(p+q) + 4pq)/dq = -2 + 4p = 0 \)
  - (also for player 2)

No better result achievable for \( p=q=0.5 \)

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Nash Equilibrium for \( p=q=0.5 \)!
While there need not be a Nash equilibrium for the pure strategies, there is always at least one Nash equilibrium for mixed strategies.
How to solve this a bit more easily?

Consider *matching pennies* again:

- If Player 2 chooses a probability of $q$, and Player 1 chooses Head the expected payoff to Player 1 is:
  - $(-1)q + 1(1-q) = 1-2q$
- If Player 1 chooses Tail:
  - $1q + (-1)(1-q) = 2q-1$

- Note 1: no pure strategy can be part of a Nash equilibrium
- Note 2a: If $1-2q > 2q-1$ then Head is the best strategy for P1
- Note 2b: If $1-2q < 2q-1$ then Tail is the best strategy for P1
- Note 2c: Pure strategies do not lead to Nash equilibrium, thus Player 2 must choose a point where $1-2q = 2q-1$
- Note 2d: Likewise for Player 1
Derived Principle

- A mixed equilibrium arises when the probabilities used by each player make his opponent indifferent between his two options.
Applying the „indifference principle“ to another example

„American Football“

Assume defense chooses a probability of q for defending against the pass:

❖ Then payoff for offense from passing is
  • 0 q + 10 (1-q) = 10-10q

❖ Payoff from running is
  • 5q + 0 (1-q) = 5q

❖ 10-10q=5q is solved for q=2/3

❖ Likewise p=1/3

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<tr>
<th>Offense\Defense</th>
<th>Defend Pass</th>
<th>Defend Run</th>
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<tr>
<td>Pass</td>
<td>0,0</td>
<td>10,-10</td>
</tr>
<tr>
<td>Run</td>
<td>5,-5</td>
<td>0,0</td>
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Multiple Nash equilibria

- How many Nash equilibria in this case?

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Unbalanced coordination game
Multiple Nash equilibria

- Mixed-strategy equilibrium
  - \[ 1 \cdot q + 0 \cdot (1-q) = 0 \cdot q + 2 \cdot (1-q) \]
  - \[ \rightarrow q = \frac{2}{3} \]

  Also, because of symmetry
  - \[ \rightarrow p = \frac{2}{3} \]

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Unbalanced coordination game
ZERO SUM GAMES AND LINEAR OPTIMIZATION
Linear Programming

- Linear optimization task
  - Maximize or minimize a linear objective function in multiple variables
    \[
    \max c_1 x_2 + \ldots + c_n x_n = c^T x \quad c, x \in \mathbb{R}^n
    \]

- Linear constraints
  - Given with linear (in-)equalities
    \[
    a_{11} x_1 + \ldots + a_{1n} x_n \leq b_1 \\
    \vdots \quad \iff \quad A \in \mathbb{R}^{mxn}, b \in \mathbb{R}^m, x \in \mathbb{R}^n \\
    a_{m1} x_1 + \ldots + a_{mn} x_n \leq b_m \\
    x_i \geq 0, \forall i = 1 \ldots n
    \]
### Zero sum game

Players x and y choose election topics to gather votes. Depending on the chosen topics they gain or loose votes. What is the best mixed strategy?

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</tr>
<tr>
<td><strong>Social Welfare</strong></td>
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<td>1</td>
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Zero Sum Game

Player \( x \) knows that player \( y \) tries to minimize his loss by minimizing the following two values:

\[
\min \{3x_1 - 2x_2, -x_1 + x_2\}
\]

While \( x \) tries to maximize this value

\[
\max \min \{3x_1 - 2x_2, -x_1 + x_2\}
\]

Dual argument for player \( y \):

\[
\min \max \{3y_1 - y_2, -2y_1 + y_2\}
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Write zero sum game as linear programs

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<td>max min {3x_1 - 2x_2, -x_1 + x_2}</td>
<td>min max {3y_1 - y_2, -2y_1 + y_2}</td>
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max z

\[
\begin{align*}
3x_1 - 2x_2 &\geq z \\
-x_1 + x_2 &\geq z \\
x_1 + x_2 &= 1 \\
x_1, x_2 &\geq 0
\end{align*}
\]

min w

\[
\begin{align*}
3y_1 - y_2 &\leq w \\
-2y_1 + y_2 &\leq w \\
y_1 + y_2 &= 1 \\
y_1, y_2 &\geq 0
\end{align*}
\]
Dual Linear Programs

Every linear programming problem, referred to as a *primal* problem, can be converted into a *dual problem*:

- Given primal problem:
  Maximize $\mathbf{c}^T \mathbf{x}$ subject to $A \mathbf{x} \leq \mathbf{b}$, $\mathbf{x} \geq 0$;

- Has same value of the objective function as the corresponding dual problem:
  Minimize $\mathbf{b}^T \mathbf{y}$ subject to $A^T \mathbf{y} \geq \mathbf{c}$, $\mathbf{y} \geq 0$. 
Write zero sum game as linear programs

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y_1, y_2 \geq 0 & \\
\end{array} \] |

Observation:
The two problems are dual to each other, thus they have the same objective function value: Equilibrium
Minimax Theorem

- Minimax-Theorem:

\[
\max_x \min_y \sum_j x_i y_j G_{i,j} = \min_y \max_x \sum_i x_i y_j G_{i,j}
\]

- There are mixed strategies that are optimal for both players.
Zero Sum Game

What are the best strategies now?

- Player x chooses strategies with $x = \left(\frac{3}{7}, \frac{4}{7}\right)$
- Player y chooses strategies with $y = \left(\frac{2}{7}, \frac{5}{7}\right)$
- Both achieve an objective function value of $\frac{1}{7}$
PARETO OPTIMALITY & SOCIAL OPTIMALITY
### Optimality – 1

- **Nash Equilibrium**
  - Individual optimum (-4, -4)
  - Not a global optimum

#### Prisoners' Dilemma

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Social Optimality

**Definition**: A choice of strategies – one by each player – is a *social welfare maximizer* (or *socially optimal*) if it maximizes the sum of the players’ payoffs.

However: Adding payoffs does not always make sense!

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**Prisoners’ Dilemma**
Pareto Optimality

Definition: A choice of strategies – one by each player – is **Pareto-optimal** if there is no other choice of strategies in which all players receive payoffs at least as high, and at least one player receives a strictly higher payoff.

- Core here: **binding agreement**
- (not-confess, not-confess) is Pareto-optimal, but not a Nash equilibrium
- **Nash equilibrium is only Non-Pareto optimal choice!**

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MULTIPLAYER GAMES
Structure:

- n players
- each player has set of possible strategies
- each player has a payoff function $P_i$ such that for reach outcome consisting of strategies chosen by the players $(S_1, S_2, \ldots S_n)$, there is a payoff $P_i(S_1, S_2, \ldots S_n)$ to player i.

**Best response**: A strategy $S_i$ is a *best response* by Player i to a choice of strategies $(S_1, S_2, \ldots S_{i-1}, S_{i+1}, \ldots, S_n)$, if

$$P_i(S_1, \ldots S_{i-1}, S_i, S_{i+1}, \ldots, S_n) \geq P_i(S_1, \ldots S_{i-1}, S'_i, S_{i+1}, \ldots, S_n)$$

for all other possible strategies $S'_i$ available to player i.
Multiplayer Games

Structure:

- n players
- each player has set of possible strategies
- each player has a payoff function $P_i$ such that for each outcome consisting of strategies chosen by the players $(S_1, S_2, \ldots, S_n)$, there is a payoff $P_i(S_1, S_2, \ldots, S_n)$ to player $i$.

Nash equilibrium: A choice of strategies $(S_1, S_2, \ldots, S_n)$ is a Nash equilibrium if each strategy it contains is a best response to all others.
DOMINATED STRATEGIES
**Dominated Strategies**

**Definition**: A strategy is *strictly dominated* if there is some other strategy available to the same player that produces a strictly higher payoff in response to every choice of strategies by the other players.

(notion is only useful if there are more than two strategies)
Facility Location Game (cf Hotelling’s law)

- Firm 1: Strategy A is dominated by C
- Firm 2: Strategy F is dominated by D
- Reason about the structure and delete these strategies
Facility Location Game (cf Hotelling’s law)

- Firm 1: Strategy E is dominated by C
- Firm 2: Strategy B is dominated by D
- Reason about the structure and delete these strategies
Facility Location Game (cf Hotelling’s law)

- One pair of strategies remaining
- The unique Nash equilibrium
- Generalizes from 6 to an arbitrary number of nodes